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CERTAIN RESULTS OF ANALYTIC FUNCTION ASSOCIATED WITH CONIC SECTIONS AND QUASI SUBORDINATION USING q-DERIVATIVE

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Abstract. Using the notion of quasi-subordination, we presumably define the new classes of analytic functions of complex order involving Hadamard factorization. Further, the classes are defined by replacing the ordinary differential with q-difference operator. Initial coefficient bounds and the Fekete-Szegő inequality for the classes associated with the conic section has been obtained. Further, several applications of our results have been established.

Keywords: univalent function, error function, subordination, conic sections, quasi-subordination, q-derivative operator.

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1. Introduction.

Let A denote the class of function analytic in the open unit disk $\Upsilon = \{ z \in \mathrm{X} : |z| < 1 \}$ with Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

and Σ be the class of functions in A which are univalent in Υ. Here Ω denotes the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots, \tag{1.2}$$

analytic and satisfying a condition $|w(z)| < 1$ in Υ, known as a class of Schwarz functions. To recall the principle of subordination between analytic function let the functions f and g be analytic in Υ. Then we say that the function f is subordinate to g, if there exists a Schwarz function w, such that $f(z) = g(w(z)), (z \in \Upsilon)$. We denote this subordination by $f \prec g$ (or $f(z) \prec g(z), z \in \Upsilon$). In particular, if the function g is univalent in Υ, the above subordination is equivalent to the conditions $f(0)=g(0), f(\Upsilon)\subset g(\Upsilon).$ The notion of the subordination was extended to quasi-subordination by Robertson in [29]. We call a function f quasi-subordinate to a function g in Υ if there exist the Schwarz function w and an analytic functio φ satisfying $|\varphi(z)|<1$ such that $f(z)=\varphi(z)g(w(z))$ in Υ. We then write $f \prec_q g$. If $\varphi(z)\equiv 1$ then the quasi-subordination reduces to the subordination. If we set $w(z)=z$, then $f(z)=\varphi(z)g(z)$ and we say that f is majorized by g and it is written as $f(z)= \;\; g(z)$ in Υ. Therefore quasi-subordination is a generalization of the notion of the subordination as well as the majorization that underline its importance. Related works of quasi-subordination may be found in [12]. Let $f\in \mathrm{A}$ be given by (1.1) and g be given by

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n. \tag{1.3}$$

The convolution or Hadamard product of $f(z)$ and $g(z)$ is denoted by $(f*g)$ and is defined as

$$(f*g)(z)=\sum_{n=2}^{\infty}a_n b_n z^n. \tag{1.4}$$

Quantum calculus which is very famously called as q-calculus is a idea which was based on the method of finite difference rescaling. The difference of quantum differentials from the ordinary ones is that notion of limit is removed in q-calculus. The application of $q-$calculus was initiated by Jackson [14, 15]. He was the first mathematician who developed $q-$derivative and $q-$integral in a systematic way. Purohit and Raina [25], Kanas and Răducanu [23] have used the fractional $q-$calculus operators in investigations of certain classes of functions which are analytic in the open disk. A comprehensive study on applications of $q-$calculus in operator theory may be found in [5]. Both operators play crucial role in the theory of relativity, usually encompasses two theories by Einstein, one in special relativity and the other in general relativity. Special relativity applies to the elementary particles and their interactions, whereas general relativity applies to the cosmological and astrophysical realm, including astronomy. Special relativity theory rapidly became a significant and necessary tool for theorists and experimentalists in the new fields of atomic physics, nuclear physics and quantum mechanics. The $q-$difference operator denoted as $D_q f(z)$ is defined by

$$D_q f(z)=\begin{cases}\dfrac{f(z)-f(qz)}{z(1-q)} & for \quad z\neq 0,\\ f'(0) & for \quad z=0,\end{cases} \tag{1.5}$$

and $D_q^2 f(z) = D_q\left(D_q f(z)\right)$.
From (1.5), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q}. \tag{1.6}$$

As $q \to 1^-, [n]_q \to n$. For a function $h(z) = z^n$, we observe that

$$D_q(h(z)) = D_q(z^n) = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1},$$

$$\lim_{q\to 1} D_q(h(z)) = \lim_{q\to 1}\left([n]_q z^{n-1}\right) = nz^{n-1} = h'(z),$$

where h' is the ordinary derivative.
As a right inverse, Jackson [14] introduced the $q-$integral

$$\int_0^z h(t) d_q t = z(1-q)\sum_{n=0}^{\infty} q^n f\left(zq^n\right),$$

provided that the series converges. For a function $h(z) = z^n$, we observe that

$$\int_0^z h(t) d_q t = \lim_{q\to 1^-} \frac{z^{n+1}}{[n+1]_q} = \frac{z^{n+1}}{n+1} = \int_0^z h(t) dt,$$

where $\int_0^z h(t) dt$ is the ordinary integral. In the sequel we will use $q-$operators to the functions related to the conic sections, that were introduced and studied by Kanas et al. [16]–[22] and examinated by several mathematicians in a series of papers, see for example Kanas and Raducanu [23], Sim et al. [30], etc. Kharasani [3] extended original definition to the $p-$valent functions generalizing the domains Ω_k to $\Omega_{k,\alpha}$ $(0 \le k < \infty, 0 \le \alpha < 1)$ as follows:

$$\Omega_{k,\alpha} = \left\{w = u + iv : (u-\alpha)^2 > k^2(u-1)^2 + k^2 v^2\right\}, \quad \Omega_{k,0} = \Omega_k.$$

Various classes of functions were defined by the fact of membership to the domain $\Omega_{k,\alpha}$, for instance by setting $w = p(z) = \frac{z f'(z)}{f(z)}$ or $p(z) = 1 + \frac{z f''(z)}{f'(z)}$. We note that the explicit form of function $p_{k,\alpha}$ that maps the unit disk onto the domains bounded by $\Omega_{k,\alpha}$ and such that $1 \in \Omega_{k,\alpha}$ is as follows

$$p_{0,\alpha}(z) = \frac{1+(1-2\alpha)z}{1-z}, \quad p_{1,\alpha}(z) = 1 + \frac{2(1-\alpha)}{\pi^2} \log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}},$$

$$p_{k,\alpha}(z) = \begin{cases} \dfrac{1-\alpha}{1-k^2} \cos\left(A(k) i \log \dfrac{1+\sqrt{z}}{1-\sqrt{z}} \right) - \dfrac{k^2-\alpha}{1-k^2} & 0 < k < 1, \\ \dfrac{1-\alpha}{k^2-1} \sin^2\left(\dfrac{\pi}{2k(t)} \mathrm{K}\left(\dfrac{\sqrt{z}}{\sqrt{t}}, t \right) \right) + \dfrac{k^2-\alpha}{k^2-1} & k > 1, \end{cases} \tag{1.7}$$

with $t \in (0,1)$ chosen such that $k = \cosh \dfrac{\pi k'(t)}{4k(t)}$.

By virtue of the properties of the domains, for $p \prec p_{k,\alpha}$, we have

$$\mathrm{R}p(z) \geq \mathrm{R}(p_{k,\alpha}(z)) > \frac{k+\alpha}{k+1}.$$

Note that Kanas and Sugawa [20] proved the positivity of coefficients of the functions $p_{k,0}$ that implies positivity of $p_{k,\alpha}$ for $0 \leq \alpha < 1$. Also, we note that the domains $\Omega_{k,\alpha}$ are symmetric about real axis and starlike with respect to 1.

The error function erf defined by [1]

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!}, \tag{1.8}$$

is the subject of intensive studies and applications during the last years. Several properties and inequalities of error function can be found in [4, 8]. In [10] the authors study the properties of complementary error function occurring widely in almost every branch of applied mathematics and mathematical physics, e.g., probability and statistics [7] and data analysis [13]. Its inverse, introduced by Carlitz [6], which we will denote by $inver\,f$, appears in multiple areas of mathematics and the natural sciences.

Let $E_r f$ (see [28]) be a normalized analytic function which is obtained from (1.1), and given by

$$E_r f = z + \sum_{j=2}^{\infty} \frac{(-1)^{n-1} z^n}{(2n-1)(n-1)!}.$$

Motivated by [12, 17], we define the following classes.

Definition 1. *Let* $0 < q < 1$, $\gamma \in \mathrm{X}\backslash\{0\}$, *and let* $p_{k,\alpha}(z)$ *be defined as in (1.7). The functions* $f, g \in \mathsf{A}$ *are in the class* $\vartheta\Sigma_g\left(q, \gamma, (p_{k,\alpha})\right)$ *if*

$$1 - \frac{2}{\gamma} + \frac{2}{\gamma}\left(\frac{zD_q(f*g)(z)}{(f*g)(z)}\right) \prec p_{k,\alpha}(z) \quad (z \in \mathrm{Y}). \tag{1.9}$$

Simillarly a function $f \in \mathrm{A}$ is in the class $\vartheta\mathrm{X}_g\left(q, \gamma, (p_{k,\alpha})\right)$ if

$$1 - \frac{2}{\gamma} + \frac{2}{\gamma}\left(\frac{D_q\left(zD_q(f*g)(z)\right)}{D_q(f*g)(z)}\right) \prec p_{k,\alpha}(z) \quad (z \in \mathrm{Y}).$$

Let $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots (c_1 > 0)$ be an analytic function with positive real part on Y which maps the open disk Y on to a region starlike with respect to 1 and symmetric with respect to the real axis. Let $\varphi(z) = d_0 + d_1 z + d_2 z^2 + \cdots$ and $|d_n| \leq 1$.

Definition 2. *Let* $0 < q < 1$, $\gamma \in \mathrm{X}\backslash\{0\}$. *By* $\vartheta\Sigma_g\left(q, \gamma, \phi\right)$ *we mean a family that consist of the functions* $f, g \in \mathsf{A}$ *satisfying quasi-subordination*

$$\frac{1}{\gamma}\left(\frac{zD_q(f*g)(z)}{(f*g)(z)} - 1\right) \prec_q \phi(z) - 1 \quad (z \in \mathrm{Y}),$$

and let the class $\vartheta\mathrm{X}_g\left(q, \gamma, \phi\right)$ consist of the functions $f \in \mathrm{A}$ satisfying quasi-subordination

$$\frac{1}{\gamma}\left(\frac{D_q\left(zD_q(f*g)(z)\right)}{D_q(f*g)(z)} - 1\right) \prec_q \phi(z) - 1 \quad (z \in \mathrm{Y}).$$

The principal significance of the sharp bounds of the coefficients is the information about geometric properties of the functions. For instance, the sharp bounds of the second coefficient of normalized univalent functions readily yields the growth and distortion bounds. Also, sharp bounds of the coefficient functional $|a_3 - \mu a_2^2|$ obviously help in the investigation of univalence of analytic functions. Apart from these n-th coefficient bounds were used to determine the extreme points of the classes of analytic functions. Estimates of Fekete-Szegő functional for various subclasses of univalent and multivalent functions were given, among other, in [9, 26, 27].

In this paper, we obtain coefficient estimates for the functions in the above defined class for $q-$difference operator associated with subordination and quasi subordination.

The following lemma regarding the coefficients of functions in Ω are needed to prove our main results.

Lemma 1. *[2] If* $w \in \Omega$ *then* $|w_2 - tw_1^2| \le max\{1,|t|\}$, for any complex number t. The result is sharp for the function $w(z) = z$ or $w(z) = z^2$.

2. The Fekete-Szegő functional associated with the conic sections.

In this section we will consider the behaviour of the Fekete-Szegő functional defined on the classes related to the conical domains.

Theorem 1. *Let* $0 \le k < \infty$, $0 \le \alpha < 1$, $0 < q < 1$, $\gamma \in \mathrm{X} \backslash \{0\}$ *and let* $p_{k,\alpha}(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *and* $f \in \mathcal{G}\Sigma_g(q,\gamma,(p_{k,\alpha}))$. *Then for any complex number* μ,

$$|a_3 - \mu a_2^2| \le \frac{|\gamma|\, p_1}{2|b_3|\rho_3} max\left\{1, \left|\frac{p_2}{p_1} + \frac{b_2^2\rho_2 - \mu b_3\rho_3}{2b_2^2\rho_2^2}\gamma p_1\right|\right\}, \tag{2.1}$$

where $\rho_2 = [2]_q - 1$ and $\rho_3 = [3]_q - 1$.

Proof. If $f \in \mathcal{G}\Sigma_g(q,\gamma,(p_{k,\alpha}))$, then there exist a Schwarz function $w \in \Omega$ of the form (1.2) such that

$$1 - \frac{2}{\gamma} + \frac{2}{\gamma}\left(\frac{zD_q(f*g)(z)}{(f*g)(z)}\right) \prec p_{k,\alpha}(w(z)) \quad (z \in \mathrm{Y}). \tag{2.2}$$

We note that

$$\frac{zD_q(f*g)(z)}{(f*g)(z)} = 1 + ([2]_q - 1)a_2 b_2 z + (([3]_q - 1)a_3 b_3 + (1 - [2]_q)a_2^2 b_2^2)z^2 + \cdots \tag{2.3}$$

and

$$p_{k,\alpha}(w(z)) = 1 + p_1 w_1 z + (p_1 w_2 + p_2 w_1^2)z^2 + (p_1 w_3 + 2p_2 w_1 w_2 + p_3 w_1^3)z^3 + \cdots. \tag{2.4}$$

Applying (2.2), (2.3) and (2.4), we obtain

$$a_2 = \frac{\gamma p_1 w_1}{2b_2([2]_q - 1)} \tag{2.5}$$

and

$$a_3 = \frac{\gamma p_1}{2b_3([3]_q - 1)}\left[w_2 - \left(\frac{-p_2}{p_1} - \frac{p_1\gamma}{2([2]_q - 1)}\right)w_1^2\right]. \tag{2.6}$$

Hence, by (2.5) and (2.6), we get the following

$$a_3 - \mu a_2^2 = \frac{\gamma p_1}{2([3]_q - 1)b_3}(w_2 - t w_1^2), \qquad (2.7)$$

where

$$t = -\frac{p_2}{p_1} - \left(\frac{b_2^2([2]_q - 1) - \mu b_3([3]_q - 1)}{2b_2^2([2]_q - 1)^2}\right)\gamma p_1.$$

The result (2.1) is established by an application of Lemma 1.

Theorem 2. *Let* $0 \le k < \infty, 0 \le \alpha < 1, 0 < q < 1,\ \gamma \in \mathrm{X}\setminus\{0\}$ *and let* $p_{k,\alpha}(z) = 1 + p_1 z + p_2 z^2 + \cdots$. *For* $f \in \mathcal{G}\mathrm{X}_g(q,\gamma,(p_{k,\alpha}))$ *and for any complex number* μ, *it holds*

$$|a_3 - \mu a_2^2| \le \frac{|\gamma| p_1}{2|b_3|[3]_q \rho_3} \max\left\{1, \left|\frac{p_2}{p_1} + \frac{b_2^2[2]_q^2 \rho_2 - \mu b_3 [3]_q \rho_3}{2b_2^2[2]_q^2 \rho_2^2}\gamma p_1\right|\right\}, \qquad (2.8)$$

where $\rho_2 = [2]_q - 1$ and $\rho_3 = [3]_q - 1$.

The result (2.8) is established by an application of Lemma 1.

If g(z)= $E_r f = z + \sum_{j=2}^{\infty} \frac{(-1)^{n-1} z^n}{(2n-1)(n-1)!}$ in the above Theorems 1 & 2 then we get the following corollaries, which are closely related to the results by Ramachandran ([28], Theorem 2.1).

Corollary 1. *Let* $0 \le k < \infty, 0 \le \alpha < 1, 0 < q < 1,\ \gamma \in \mathrm{X}\setminus\{0\}$ *and let* $p_{k,\alpha}(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *and* $f \in \mathcal{G}\Sigma(q,\gamma,(p_{k,\alpha}))$. *Then for any complex number* μ,

$$|a_3 - \mu a_2^2| \le \frac{5|\gamma| p_1}{[3]_q - 1} \max\left\{1, \left|\frac{p_2}{p_1} + \frac{10\rho_2 - 9\mu\rho_3}{20\rho_2^2}\gamma p_1\right|\right\}, \qquad (2.9)$$

where $\rho_2 = [2]_q - 1$ and $\rho_3 = [3]_q - 1$.

Corollary 2. *Let* $0 \le k < \infty, 0 \le \alpha < 1, 0 < q < 1,\ \gamma \in \mathrm{X}\setminus\{0\}$ *and let* $p_{k,\alpha}(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *and* $f \in \mathcal{G}\mathrm{X}(q,\gamma,(p_{k,\alpha}))$. *Then for any complex number* μ,

$$|a_3 - \mu a_2^2| \le \frac{5|\gamma| p_1}{[3]_q([3]_q - 1)} \max\left\{1, \left|\frac{p_2}{p_1} + \frac{10[2]_q^2 \rho_2 - 9\mu[3]_q \rho_3}{20[2]_q^2 \rho_2^2}\gamma p_1\right|\right\}, \qquad (2.10)$$

where $\rho_2 = [2]_q - 1$ and $\rho_3 = [3]_q - 1$.

3.The Fekete-Szegő functional associated with the quasi-subordination.

Theorem 3. *Let* $0 < q < 1$, $\gamma \in \mathrm{X} \setminus \{0\}$. *If* f *is of the form (1.1) belongs to* $\vartheta\Sigma_g(q,\gamma,\phi)$*, then*

$$|a_2| \le \frac{|\gamma| c_1}{([2]_q - 1)|b_2|}, \tag{3.1}$$

$$|a_3| \le \frac{|\gamma|}{([3]_q - 1)|b_3|}\left(c_1 + max\left\{ c_1, \frac{|\gamma| c_1^2}{[2]_q - 1} + |c_2| \right\} \right), \tag{3.2}$$

and for any complex number μ,

$$|a_3 - \mu a_2^2| \le \frac{|\gamma|}{([3]_q - 1)|b_3|}\left(c_1 + max\left\{ c_1, \left| \frac{([2]_q - 1)b_2^2 - \mu([3]_q - 1)b_3}{([2]_q - 1)^2 b_2^2} \right| |\gamma| c_1^2 + |c_2| \right\} \right). \tag{3.3}$$

Proof. If $f \in \vartheta\Sigma_g(q,\gamma,\phi)$ then there exists analytic functions φ and w with $|\varphi(z)| \le 1$, $w(0) = 0$ and $|w(z)| < 1$ such that,

$$\frac{1}{\gamma}\left(\frac{zD_q(f * g)(z)}{(f * g)(z)} - 1 \right) = \varphi(z)(\phi(w(z) - 1)), \tag{3.4}$$

$$\frac{1}{\gamma}\left(\frac{zD_q(f * g)(z)}{(f * g)(z)} - 1 \right) = \frac{1}{\gamma}\left[([2]_q - 1)a_2 b_2 z + (([3]_q - 1)a_3 b_3 + (1 - [2]_q)a_2^2 b_2^2)z^2 + \cdots \right], \tag{3.5}$$

$$\varphi(z)(\phi(w(z) - 1)) = c_1 d_0 w_1 z + (c_1 d_1 w_1 + d_0(c_1 w_2 + c_2 w_1^2))z^2 + \cdots. \tag{3.6}$$

From (3.4), (3.5) and (3.6), we get

$$a_2 = \frac{\gamma c_1 d_0 w_1}{([2]_q - 1)b_2},$$

and

$$a_3 = \frac{\gamma}{([3]_q - 1)b_3}\left(c_1 d_1 w_1 + c_1 d_0 w_2 + d_0\left(c_2 + \frac{\gamma c_1^2 d_0}{[2]_q - 1} \right) w_1^2 \right).$$

Since $\varphi(z)$ is analytic and bounded in Y, we have ([24], pg 172) $|d_n| \le 1 - |d_0|^2 \le 1 (n > 0)$.

By using the fact $|d_n| \le 1$ and $|w_1| \le 1$, we get

$$|a_2| \le \frac{|\gamma| c_1}{([2]_q - 1)|b_2|},$$

$$|a_3 - \mu a_2^2| \le \frac{|\gamma| c_1}{([3]_q - 1)|b_3|}(1 + |w_2 - t w_1^2|),$$

where

$$t = -\left[\frac{\gamma}{[2]_q - 1} - \frac{\mu\gamma([3]_q - 1)b_3}{([2]_q - 1)^2 b_2^2}\right] c_1 d_0 - \frac{c_2}{c_1}.$$

Applying Lemma 1, we get

$$|a_3 - \mu a_2^2| \le \frac{|\gamma|}{([3]_q - 1)|b_3|}\left(c_1 + \max\left\{c_1, \left|\frac{\gamma}{[2]_q - 1} - \frac{\mu\gamma([3]_q - 1)b_3}{([2]_q - 1)^2 b_2^2}\right| c_1^2 + |c_2|\right\}\right).$$

For $\mu = 0$, we get

$$|a_3| \le \frac{|\gamma|}{([3]_q - 1)|b_3|}\left(c_1 + \max\left\{c_1, \frac{|\gamma| c_1^2}{[2]_q - 1} + |c_2|\right\}\right).$$

Theorem 4. *Let* $0 < q < 1$, $\gamma \in \mathrm{X}\backslash\{0\}$. *If* f *is of the form (1.1) belongs to* $\vartheta\mathrm{X}_g(q,\gamma,\phi)$, *then*

$$|a_2| \le \frac{|\gamma| c_1}{[2]_q([2]_q - 1)|b_2|},$$

$$|a_3| \le \frac{|\gamma|}{[3]_q([3]_q - 1)|b_3|}\left(c_1 + \max\left\{c_1, \frac{|\gamma| c_1^2}{[2]_q - 1} + |c_2|\right\}\right),$$

and for any complex number μ,

$$|a_3 - \mu a_2^2| \le \frac{|\gamma|}{[3]_q([3]_q - 1)|b_3|} \times$$

$$\times\left(c_1 + \max\left\{c_1, \left|\frac{[2]_q^2([2]_q - 1)b_2^2\gamma - \mu[3]_q([3]_q - 1)b_3}{[2]_q^2([2]_q - 1)^2 b_2^2}\right| |\gamma| c_1^2 + |c_2|\right\}\right).$$

Letting $q \to 1$, and $g(z) = \dfrac{z}{1 - z}$ in Theorem 3, we obtained the following corollary.

Corollary 3. *[9] If* $f(z)$ *given by (1.1) belongs to the class* $\Sigma_q^*(\gamma,\phi)$ *and* μ *is a complex number, then*

$$|a_2| \leq |\gamma| c_1,$$

$$|a_3| \leq \frac{|\gamma|}{2}\left(c_1 + max\{c_1, |\gamma| c_1^2 + |c_2|\}\right),$$

and

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma|}{2}\left(c_1 + max\{c_1, |\gamma||1-2\mu| c_1^2 + |c_2|\}\right).$$

For $\gamma = 1$, Corollary 3 reduces to the following corollary.

Corollary 4. *[12] Let* $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots, c_1 > 0$ *and* $\varphi(z) = B_0 + B_1 z + B_2 z^2 + \cdots$. *If* $f(z)$ *given by (1.1) belongs to the class* $\Sigma_q^*(\phi)$ *and* μ *is a complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{2}\left(c_1 + max\{c_1, |1-2\mu| c_1^2 + |c_2|\}\right).$$

Letting $q \to 1$, and $g(z) = \dfrac{z}{1-z}$ in Theorem 4, we obtained the following corollary.

Corollary 5. *[9] If* $f(z)$ *given by (1.1) belongs to the class* $X_q^*(\gamma,\phi)$ *and* μ *is a complex number, then*

$$|a_2| \leq \frac{|\gamma| c_1}{2},$$

$$|a_3| \leq \frac{|\gamma|}{6}\left(c_1 + max\{c_1, |\gamma| c_1^2 + |c_2|\}\right),$$

and

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma|}{6}\left(c_1 + max\left\{c_1, \frac{|\gamma||2-3\mu|}{2} c_1^2 + |c_2|\right\}\right).$$

For $\gamma = 1$, Corollary 5 reduces to the following corollary.

Corollary 6. *[12] Let* $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots, c_1 > 0$ *and* $\varphi(z) = B_0 + B_1 z + B_2 z^2 + \cdots$. *If* $f(z)$ *given by (1.1) belongs to the class* $X_q(\phi)$ *and* μ *is a complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{6}\left(c_1 + max\left\{c_1, \left|1 - \frac{3}{2}\mu\right| c_1^2 + |c_2|\right\}\right).$$

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